

**FRACTIONAL CALCULUS APPLIED IN SOLVING
INSTABILITY PHENOMENON IN FLUID DYNAMICS**

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ABSTRACT

The purpose of present paper is to find applications of Fractional Calculus approach in Fluid Mechanics. In this paper by generalizing the instability phenomenon in fluid flow through porous media with mean capillary pressure with transforming the problem into Fractional partial differential equation and solving it by applying Fractional Calculus and special functions.

Keywords: Fluid flow through porous media, Laplace transform, Fourier sine transform, Mittag - Leffler function, Fox-Wright function, Fractional time derivative.

1. INTRODUCTION AND MATHEMATICAL PREREQUISITES

Using ideas of ordinary calculus, we can differentiate a function, say, $f(x)=x$ to the 1st or 2nd order. We can also establish a meaning or some potential applications of the results. However, can we differentiate the same function to, say, the 1/2 order? Better still, can we establish a meaning or some potential applications of the results? We may not achieve that through ordinary calculus, but we may through fractional calculus—a more generalized form of calculus. Fractional Calculus is the branch of calculus that generalizes the derivative of a function to non-integer order. In other words the fractional calculus operators deal with integrals and derivatives of arbitrary (i.e. real or complex) order. The name "fractional calculus" is actually a misnomer; the designation, "integration and differentiation of arbitrary order" is more appropriate. The subject calculus independently discovered in 17th century by Isaac Newton and Gottfried Wilhelm Leibnitz, the question raised by Leibnitz for a fractional derivative was an ongoing topic for more than three hundred years. For a long time, fractional calculus has been regarded as a pure mathematical realm without real applications. But, in recent decades, such a state of affairs has been changed. It has been found that fractional calculus can be useful and even powerful, and an outline of the simple history about fractional calculus.. The various researcher investigated their investigations dealing with the theory and applications of

fractional calculus Free shear flows are inhomogeneous flows with mean velocity gradients that develop in the absence of cigarette, and the buoyant jet issuing from an erupting volcano - all illustrate both the omnipresence of free turbulent shear flows and the range of scales of such flows in the natural environment. Examples of the multitude of engineering free shear flows are the wakes behind moving bodies and the exhausts from jet engines. Most combustion processes and many mixing processes involve turbulent free shear flows. Free shear flows in the real world are most often turbulent. The tendency of free shear flows to become and remain turbulent can be greatly modified by the presence of density gradients in the flow, especially if gravitational effects are also important. Free share flows deals with incompressible constant-density flows away from walls, which include shear layers, jets and wakes behind bodies. Hydrodynamic stability is of fundamental importance in fluid dynamics and is a well-established subject of scientific investigation that continues to draw great curiosity of the fluid mechanics community. Hydrodynamic instabilities of prototypical character are, for example, the Rayleigh-Bènard, the Taylor-Couette, the Bènard-Marangoni, the Rayleigh-Taylor, and the Kelvin-Helmholtz instabilities. Modeling of various instability mechanisms in biological and biomedical systems is currently a very active and rapidly developing area of research with important biotechnological and medical applications (biofilm engineering, wound healing, etc.). The understanding of breaking symmetry in hemodynamics could have important consequences for vascular biology and diseases and its implication for vascular interventions (grafting, stenting, etc.). When in a porous medium filled with one fluid and another fluid is injected which is immiscible in nature in ordinary condition, then instability occurs in the flow depending upon viscosity difference in two flowing phases. When a fluid flow through porous medium displaced by another fluid of lesser viscosity then instead of regular displacement of whole front protuberance take place which shoot through the porous medium at a relatively high speed. This phenomenon is called fingering phenomenon (or instability phenomenon). Many researchers have studied this phenomenon with different point of view. Fractional calculus is now considered as a practical technique in many branches of science including physics (Oldham and Spainier [13]). A growing number of works in science and engineering deal with dynamical system described by fractional order equations that involve derivatives and integrals of non-integer order (Benson et al. [2], Metzler and Klafter [9], Zaslavsky [24]). These new models are more adequate than the previously used Integer order models, because fractional order derivatives and integrals describe the memory and hereditary properties of different substances (Podlubny [14]). This is the most significant advantage of the fractional order models in comparison with integer order models, in which such effects are neglected. In the context of flow in porous media, fractional space derivatives model large motions through highly conductive layers or fractures, while fractional time derivatives describe particles that remain motionless for extended period of time (Meerschaert et al. [8]).

The phenomenon of instability in polyphasic flow is playing very important role in the study of fluid flow through porous media in two ways viz. with capillary pressure and without capillary pressure. The statistical view point was studied by Scheidegger and Johnson [21], Bhathawala and Shama Parveen [3] considering instability phenomenon in porous media without mean Capillary pressure. Verma [23] has also studied the behavior of instability in a displacement process through heterogeneous porous media and existence and uniqueness of solution of the problem was discussed by Atkinson and Peletier [1]. El-Shahed and Salem [11, 12] have used the fractional calculus approach in fluid dynamics, which has been described by fractional partial differential equation and the exact solution of these equations, have been obtained by using the Laplace transform, Fourier transform. Flow in a porous medium is described by Darcy's Law (El-Shahed and Salem [11]) which relates the movement of fluid to the pressure gradients acting on a parcel of fluid. Darcy's Law is based on a series of experiments by Henry Darcy in the mid-19th century showing that the flow through a porous medium is linearly proportional to the applied pressure gradient and inversely proportional to the viscosity of the fluid. In one dimension, q represents "mass flow rate by unit area"

and is defined as,

$$q = - \frac{K}{\delta} \frac{dP}{dx}$$

where K is permeability, a parameter intrinsic to the porous network. The unit of permeability K is $\frac{m}{s}$. δ is the kinematics viscosity has dimension $L^2 T^{-1}$, e.g., $cm^2 sec^{-1}$ and $\frac{dP}{dx}$ is the non-hydrostatic part of pressure gradient has dimension $M L^{-2} T^{-2}$ e.g., $g cm^{-2} sec^{-2}$. thus the mass flow rate by unite area(q) has dimension $\frac{cm^2}{cm^2 s^{-1}} \frac{g}{cm^2 s^{-1}} = \frac{g}{cm^2 s}$ Here, we considered homogeneous dimensions but in fractional calculus dimensions are inhomogeneous.

Mathematical Prerequisites

The Laplace Transform (Sneddon [20]) is defined as,

$$\begin{aligned} L \{f(x)\} \\ = \int_0^{\infty} e^{-st} f(t) dt \quad (Re(s) > 0) \end{aligned} \quad (1.1)$$

Fourier transform of Weyl fractional derivative ${}_{-\infty} D_x^\gamma f(x, t)$ is given by (Metzler and Klafter, 2000, p 59A12, 2004)

$$F\{{}_{-\infty} D_x^\gamma f(x, t)\} = |k|^\gamma \varphi(k, t)$$

The Fourier sine transform (Debnath [5]) is given as,

$$\begin{aligned} u(n, t) \\ = \frac{2}{\sqrt{\pi}} \int_0^{\infty} u(x, t) \sin nx dx \end{aligned} \quad (1.2)$$

The Error Function ([16]) of x is given as

$$\begin{aligned} er f(x) \\ = \frac{2}{\pi} \int_0^x \exp(-t^2) dt \end{aligned} \quad (1.3)$$

And the complimentary error function of x is given as

$$\begin{aligned} er f_c(x) = \\ \frac{2}{\pi} \int_x^{\infty} \exp(-t^2) dt \end{aligned} \quad (1.4)$$

In 1903, the Swedish mathematician Gosta Mittag-Leffler [10] introduced the function $E_\alpha(z)$ which is given as ----

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad (1.5)$$

Where z is a complex variable and $\Gamma(\alpha)$ is a gamma function of α . The Mittag-Leffler function is direct generalization of the exponential function to which it reduces for $\alpha = 1$. For $0 < \alpha < 1$, $E_\alpha(z)$ interpolates between the pure exponential and a hyper geometric function $\frac{1}{1-z}$. Its significance is

realized during the last two 100 years due to its involvement in the problems of physics, chemistry, biology, engineering and applied sciences. Mittag–Leffler function naturally occurs as the solution of fractional order differential equation or fractional order integral equations. The generalization of $E_\alpha(z)$ was studied by Wiman [23] in 1905 and defined the function as

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, (\alpha, \beta \in \mathbb{C}; \operatorname{Re}(\alpha) > 0, \operatorname{Re}(\beta) > 0) \quad (1.6)$$

Which is known as Wiman's function or generalized Mittag–Leffler function as $E_{\alpha,1}(z) = E_\alpha(z)$. The Laplace transform of (1.6) takes in the following form (Shukla and Prajapati [20])

$$\begin{aligned} \int_0^{\infty} e^{-st} t^{\alpha j + \beta - 1} E_{\alpha,\beta}^{(j)}(x t^\alpha) dt \\ = \frac{j! s^{\alpha - \beta}}{(s^\alpha - x)^{j+1}} \end{aligned} \quad (1.7)$$

Where $E_{\alpha,\beta}^{(j)}(Z) = \frac{d^j}{dz^j} E_{\alpha,\beta}(z)$.

The Fox-Wright function (Craven and Csordas [4]) is given as under

$${}_p\Psi_q(x) = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^p \Gamma(a_j k + b_j)}{\prod_{j=1}^q \Gamma(c_j k + d_k)} \frac{x^k}{k!}, \quad (1.8)$$

Where $\Gamma(x)$ denotes the Gamma function of x and p and q are nonnegative integers. If we set $b_j = 1$ ($j = 1, 2, 3, \dots, p$)

and $d_j = 1$ ($j = 1, 2, 3, \dots, q$) then (1.8) reduces to the familiar generalized hyper geometric function (Craven and Csordas [4]).

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; x) = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{x^k}{k!} \quad (1.9)$$

For the study of generalized Navier - Stokes equations, El - Shahed and Salem [12] used the very special case of (1.1), given as

$$w(\alpha, \beta; z) = \sum_{j=0}^{\infty} \frac{z^j}{j! \Gamma(\alpha_j + \beta)} \quad (1.10)$$

The Laplace Transforms of (1.10) is given by

$$\begin{aligned} \int_0^{\infty} e^{-st} w(\alpha, \beta; t) dt \\ = \frac{1}{s} E_{\alpha,\beta} \left(\frac{1}{s} \right) \end{aligned} \quad (1.11)$$

The relationship between the Wright function and the Complementary Error function is given as,

$$w\left(-\frac{1}{2}, 1; z\right) = \operatorname{erfc}(z/2) \quad (1.12)$$

Riemann-Liouville fractional integrals of order μ (Khan and Abukhamash [7]) is given as
Let $f(x) \in L(a, b)$, $\mu \in \mathbb{C}$ ($\text{Re}(\mu) > 0$) then

$$\begin{aligned} {}_a I_x^\mu f(x) &= I_a^\mu f(x) \\ &= \frac{1}{\Gamma(\mu)} \int_a^x \frac{f(t)}{(x-t)^{1-\mu}} dt \quad (x > a) \end{aligned} \quad (1.13)$$

is called R-L left-sided fractional integral of order μ .
Let $f(x) \in L(a, b)$, $\mu \in \mathbb{C}$ ($\text{Re}(\mu) > 0$) then

$$\begin{aligned} {}_x I_b^\mu f(x) &= \frac{1}{\Gamma(\mu)} \int_x^b \frac{f(t)}{(t-x)^{1-\mu}} dt \quad (x < b) \end{aligned} \quad (1.14)$$

is called R-L right-sided fractional integral of order μ .

The Laplace Transform of the fractional derivative (El - Shahed and Salem [12]) is defined by

$$\int_0^\infty e^{-st} D_t^\alpha f(t) dt = s^\alpha f(s) - \sum_{j=0}^{n-1} s^{\alpha-j-1} D_t^\alpha f(0) \quad (n-1 < \alpha < n) \quad (1.15)$$

Theorem (Asymptotic expansion of Wiman function $E_{\alpha,\beta}(z)$): Let $0 < \alpha < 1$ and β be an arbitrary complex number then

$$E_{\alpha,\beta}(z) = \frac{1}{2\alpha\pi i} \int \frac{\exp\left(\epsilon^{\frac{1}{\alpha}}\right) \epsilon^{\frac{1-\beta}{\alpha}}}{\epsilon - z} d\epsilon \quad (1.16)$$

We also use following integral (El - Shahed and Salem [12]) in terms of Wright function as,

$$\int_0^\infty n \sin nx E_{\alpha,\alpha+1}(-n^2 C t^\alpha) dn = \frac{\pi}{2C t^\alpha} W\left(\frac{-\alpha}{2}, 1; \frac{-x}{\sqrt{C t^\alpha}}\right) \quad (1.17)$$

2. STATEMENT OF THE PROBLEM

If water is injected into oil saturated porous medium, then as a result perturbation (instability) occurs and develops the finger flow (Scheidegger and Johnson [19]). In this paper, our aim is to study one dimensional flow, x -indicating the direction of fluid flow with the origin at the surface, due to presence of large quantity of water at $x = 0$. We assume that water saturation at $x = 0$ is almost equal to one i.e.1 and water saturation remain constant during the displacement process. Our primary interest in this paper is to look at the possibilities of transforming the problem in form of fractional partial differential equation with proper initial and boundary conditions.

3. FORMATION OF THE PROBLEM

The seepage velocity of water (V_w) and oil (δw) are given by Darcy's law [17] as

$$= -\frac{K_w}{\delta_w} K \frac{\partial P_w}{\partial x} V_w \quad (3.1)$$

$$= -\frac{K_o}{\delta_o} K \frac{\partial P_o}{\partial x} V_o \quad (3.2)$$

And equation of continuity

$$\phi \frac{\partial S_w}{\partial t} + \frac{\partial V_w}{\partial x} = 0 \quad (3.3)$$

$$\phi \frac{\partial S_o}{\partial t} + \frac{\partial V_o}{\partial x} = 0 \quad (3.4)$$

Where K is the permeability of the homogeneous medium, Kw and Ko are the relative permeability of the water and oil, Sw and So are saturation of water and oil respectively, Pw and Po are the pressure in water and oil, phases δ_w and δ_o are the kinematics viscosities of water and oil respectively and ϕ is the porosity of medium.

For inhomogeneous dimensions, considering the Time-fractional partial differential equations of the two phases as under:

$$\phi \frac{\partial^\alpha S_w}{\partial t^\alpha} + \frac{\partial V_w}{\partial x} = 0, \quad (0 < \alpha < 1) \quad (3.5)$$

$$\phi \frac{\partial^\alpha S_o}{\partial t^\alpha} + \frac{\partial V_o}{\partial x} = 0, \quad (0 < \alpha < 1) \quad (3.6)$$

For $\alpha = 1$, equations (3.5) and (3.6) reduce to equations of continuity (3.3) and (3.4) respectively and from the definition of phase saturation [17], we have

$$S_w + S_o = 1 \quad (3.7)$$

The capillary pressure PC is defined as pressure discontinuity between the flowing phases across their common interface and assumes the function of the phase saturation is a continuous linear functional relation as

$$P_c = \beta S_w \quad (3.8)$$

$$P_c = P_o - P_w, \quad (3.9)$$

where β is constant.

Relationship between phase saturation and relative permeability [18] is given by

$$\begin{aligned} K_w &= S_w \\ K_o &= 1 - S_w \\ &= S_o \end{aligned} \quad 3.10$$

4. FORMATION OF FRACTIONAL PARTIAL DIFFERENTIAL EQUATION

Now, we put the values of V_w and V_o (from (3.1) and (3.2)) in (3.5) and (3.6) respectively, we get

$$\phi \frac{\partial^\alpha S_w}{\partial t^\alpha} = \frac{\partial}{\partial x} \left\{ \frac{K_w}{\delta_w} K \frac{\partial P_w}{\partial x} \right\} \quad (0 < \alpha < 1) \quad (4.1)$$

$$\phi \frac{\partial^\alpha S_o}{\partial t^\alpha} = \frac{\partial}{\partial x} \left\{ \frac{K_o}{\delta_o} K \frac{\partial P_o}{\partial x} \right\} \quad (0 < \alpha < 1) \quad (4.2)$$

Eliminating $\frac{\partial P_w}{\partial x}$ from (4.1) and (3.9),

$$\phi \frac{\partial^\alpha S_w}{\partial t^\alpha} = \frac{\partial}{\partial x} \left\{ K \frac{K_w}{\delta_w} \left(\frac{\partial P_o}{\partial x} - \frac{\partial P_c}{\partial x} \right) \right\} \quad (4.3)$$

From (4.2), (4.3) and (3.7) we obtained

$$\frac{\partial}{\partial x} \left[K \left\{ \frac{K_w}{\delta_w} + \frac{K_o}{\delta_o} \right\} \frac{\partial P_o}{\partial x} - \frac{K_w}{\delta_w} K \frac{\partial P_c}{\partial x} \right] = 0 \quad (4.4)$$

Integrating (4.4) with respect to x, we get

$$K \left\{ \frac{K_w}{\delta_w} + \frac{K_o}{\delta_o} \right\} \frac{\partial P_o}{\partial x} - \frac{K_w}{\delta_w} K \frac{\partial P_c}{\partial x} = -B \quad (4.5)$$

Where B is the constant of integration, whose value can be determined.

Equation (4.5) can be written as

$$\frac{\partial P_o}{\partial x} = \frac{-B}{K \frac{K_w}{\delta_w} \left\{ 1 + \frac{K_o}{K_w} \frac{\delta_w}{\delta_o} \right\}} + \frac{\frac{\partial P_c}{\partial x}}{1 + \frac{K_o}{K_w} \frac{\delta_w}{\delta_o}} \quad (4.6)$$

Substituting the value of $\frac{\partial P_o}{\partial x}$ from (4.6) in (4.3), we ar

$$\phi \frac{\partial^\alpha S_w}{\partial t^\alpha} + \frac{\partial}{\partial x} \left[\frac{K \frac{K_o}{\delta_o} \frac{\partial P_c}{\partial x}}{1 + \frac{K_o}{K_w} \frac{\delta_w}{\delta_o}} + \frac{B}{1 + \left\{ \frac{K_o}{K_w} \frac{\delta_w}{\delta_o} \right\}} \right] = 0 \quad (4.7)$$

Pressure of oil (P_o) can be written as,

$$\begin{aligned} P_o &= \frac{1}{2} (P_o + P_w) + \frac{1}{2} (P_o - P_w) \\ &= \bar{P} + \frac{1}{2} P_c \end{aligned} \quad (4.8)$$

where P is the mean pressure, which is constant.

From (4.5) and (4.8) we get,

$$B = \frac{K}{2} \left\{ \frac{K_w}{\delta_w} - \frac{K_o}{\delta_o} \right\} \frac{\partial P_c}{\partial x} \quad (4.9)$$

Substituting (4.9) in (4.7), we get

$$\begin{aligned} \phi \frac{\partial^\alpha S_w}{\partial t^\alpha} + \frac{1}{2} \frac{\partial}{\partial x} \left\{ K \frac{K_w}{\delta_w} \frac{\partial P_c}{\partial x} \frac{\partial S_w}{\partial x} \right\} \\ = 0 \quad (0 < \alpha < 1) \end{aligned} \quad (4.10)$$

Taking $K \frac{K_w}{\delta_w} \frac{\partial P_c}{\partial S_w} = -\lambda$ then (4.10) reduces in the form,

$$\frac{\partial^2 S_w}{\partial x^2} = \frac{1}{c} \frac{\partial^\alpha S_w}{\partial t^\alpha}, \quad (4.11)$$

Where $C = \frac{\lambda}{2P}$

Equation (4.11) is the desired fractional partial differential equation of motion for water saturation, which governed by the flow of two immiscible phases in a homogenous porous medium and appropriate initial and boundary conditions are associated with the description as to

$$\begin{aligned} S_w(x, 0) &= 0, \\ S_w(0, T) &= s_{w0} \\ < 1, \\ \lim_{x \rightarrow \infty} S_w(x, T) &= 0; 0 < x < \infty \end{aligned} \quad (4.12)$$

5. FORMULATION AND SOLUTION OF PROBLEM

Here we are formulating the generalized fractional partial differential equation by generalizing of Prajapati's et. al.[16] model equation which is given as it is in equation (4.11)

From (4.11), we have

$$\frac{\partial^\alpha S_w}{\partial t^\alpha} = C \frac{\partial^2 S_w}{\partial x^2}$$

Now, we **generalizing** the above equation by using fractional calculus operators in following form

$$\frac{\partial^\alpha S_w(x, t)}{\partial t^\alpha} + a \frac{\partial^\beta S_w(x, t)}{\partial t^\beta} = C \frac{\partial^\gamma S_w(x, t)}{\partial x^\gamma} + \varphi(x, t) \quad (5.1)$$

Or

$$\begin{aligned} {}_0 D_t^\alpha S_w(x, t) + a {}_0 D_t^\beta S_w(x, t) \\ = C {}_{-\infty} D_x^\gamma S_w(x, t) + \varphi(x, t) \end{aligned} \quad (5.2)$$

Where ${}_0 D_t^\alpha$ is a Riemann-Liouville Fractional Derivative, ${}_{-\infty} D_x^\gamma$ is a Weyl Fractional derivative $\varphi(x, t)$ is a constant which describes the nonlinearity in the system.

If we put $a = 0, \gamma = 2$ and $\varphi(x, t) =$ it converts into **Prajapati's et. al.[16]** paper

$$\frac{\partial^\alpha S_w(x, t)}{\partial t^\alpha} + a \frac{\partial^\beta S_w(x, t)}{\partial t^\beta} = C \frac{\partial^\gamma S_w(x, t)}{\partial x^\gamma} + \varphi(x, t) \quad (5.3)$$

Solution:- Applying Laplace transformation both sides in equation (5.2), we get

$$s^\alpha \overline{S_w}(x, s) - \sum_{r=0}^{n-1} s^{\alpha-r-1} D_t S_w(x, 0) + a s^\beta \overline{S_w}(x, s) - \sum_{r=0}^{n-1} a s^{\beta-r-1} D_t S_w(x, 0) \\ = C_{-\infty} D_t^\gamma \overline{S_w}(x, s) + \overline{f(x, s)} \quad (5.4)$$

Apply boundary conditions from (4.12)

$$s^\alpha \overline{S_w}(x, s) + a s^\beta \overline{S_w}(x, s) = C_{-\infty} D_t^\gamma \overline{S_w}(x, s) + \overline{f(x, s)} \quad (5.5)$$

As is usual, It is convenient to employ the symbol – to indicate the Laplace transform with respect to the variable t. Also $\overline{f(x, s)}$ is Laplace transform of $\varphi(x, t)$ [17].

Now we apply the Fourier transform with respect to the space variable x to the above equation both sides

$$s^\alpha \overline{S_w}^*(K, s) + a s^\beta \overline{S_w}^*(K, s) \\ = -C|K|^\gamma \overline{S_w}^*(K, s) + \overline{f(k, s)}^* \quad (5.6)$$

$$\overline{S_w}^*(K, s) = \frac{\overline{f(k, s)}^*}{s^\alpha + a s^\beta + b} \text{ where } b = C|K|^\gamma \quad (5.7)$$

$$\overline{S_w}^*(K, s) \\ = \frac{\overline{f(k, s)}^*}{s^\alpha + a s^\beta + b} \quad (5.8)$$

Applying Inverse Laplace transform both sides

$$S_w^*(K, t) = \sum_{r=0}^{\infty} (-a)^r \int_0^t \varphi^*(k, t - \varepsilon) \varepsilon^{\alpha+(\alpha-\beta)r-1} E_{\alpha, (\alpha-\beta)r+\alpha}^{r+1}(-b\varepsilon^\alpha) d\varepsilon$$

Where $b = C|K|^\gamma$ (5.9)

Now, Applying Inverse Fourier transforms both sides, we get the desired result,

$$S_w(x, t) = \sum_{r=0}^{\infty} \frac{(-a)^r}{2\pi} \int_0^t \varepsilon^{\alpha+(\alpha-\beta)r-1} \int_{-\infty}^{\infty} e^{-ikx} \varphi^*(k, t - \varepsilon) E_{\alpha, \alpha+(\alpha-\beta)r+\alpha}^{r+1}(-b\varepsilon^\alpha) dk d\varepsilon \quad (5.9)$$

Which is a new and generalized result of **Prajapati's et. al.[16]** results.

Now if we put $a=0$, $\gamma = 2$ and $\varphi = 0$

It converts into **Prajapati's et. al.[16]** results.

This is easy to write in the form of wright functions

$$S_w(x, t) = s_{w_0} W\left(-\frac{\alpha}{2}, 1; -\frac{x}{ct^\alpha}\right) \quad (5.10)$$

On setting $\alpha = 1$ and using (1.12), (5.7) reduces to,

$$S_w(x, t) = s_{w_0} \operatorname{erf}_c(x / 2\sqrt{C} t) \quad (5.11)$$

Which is same as **Prajapati's et. al.[16]** results.

CONCLUSION

The constitutive relationship model of generalization of the instability phenomenon in fluid flow through porous media with mean capillary pressure by applying fractional calculus is obtained. The exact solution of the generalized fractional partial differential equation in terms of Wright function by means of Laplace transform, Fourier transform with proper initial and boundary conditions has been found. If we set $\alpha = 1$ then equation (5.10) reduces to (5.11), this method certainly useful than conventional method as the conventional method derived only for $\alpha = 1$ (equations (3.3) and (3.4)) whose solution given by equation (5.11). While this fractional calculus together with Fourier and Laplace transforms method presented in this paper also applicable for $0 < \alpha < 1$ whose solution given by equation (5.10).

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